

The Role of $k(n)$ in the $\beta - \eta$ Bridge at the $s = 1$ Pole Boundary

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Abstract

We define the ratio coordinate $K(s) = 2^s \frac{\beta(s)}{\eta(s)}$ and its remainder $k(s) = K(s) - 2^s$ as the continuous bridge between the Dirichlet Beta function $\beta(s)$ and the Dirichlet Eta function $\eta(s)$. At the $s = 1$ pole boundary, the evaluation $\beta(1) = \pi/4$ emerges as an algebraic balance point where the $\ln(2)$ magnitude of the alternating harmonic series $\eta(1)$ cancels with the inverse logarithmic scale of the scaffold derivative.

1 The Coordinate Bridge

Let the Dirichlet Beta function and Dirichlet Eta function be defined on $\text{Re}(s) > 0$ by:

$$\beta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s}, \quad \eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} = (1 - 2^{1-s})\zeta(s) \quad (1)$$

The Coordinate Ratio $K(s)$ and Coordinate Remainder $k(s)$ are defined by:

$$K(s) = 2^s \frac{\beta(s)}{\eta(s)} \quad (2)$$

$$k(s) = K(s) - 2^s = 2^s \left(\frac{\beta(s)}{\eta(s)} - 1 \right) \quad (3)$$

Inverting this coordinate mapping expresses Dirichlet Beta directly through $\eta(s)$ and the remainder $k(s)$:

$$\beta(s) = 2^{-s} (2^s + k(s)) \eta(s) = 2^{-s} K(s) \eta(s) \quad (4)$$

2 Evaluation at the $s = 1$ Singularity

At $s = 1$:

1. The alternating harmonic series evaluates to:

$$\eta(1) = \ln(2) \approx 0.6931471806 \quad (5)$$

2. The Riemann zeta function has a simple pole with Laurent expansion $\zeta(s) = \frac{1}{s-1} + \gamma + \mathcal{O}(s-1)$, while the prefactor vanishes as:

$$(1 - 2^{1-s}) = (s-1) \ln(2) - \frac{(s-1)^2}{2} \ln^2(2) + \mathcal{O}((s-1)^3) \quad (6)$$

3. The Coordinate Ratio $K(1)$ resolves the indeterminate form $0 \cdot \infty$ via L'Hôpital's rule:

$$K(1) = \lim_{s \rightarrow 1} 2^s \frac{\beta(s)}{(1 - 2^{1-s})\zeta(s)} = 2^1 \frac{\pi/4}{\ln(2)} = \frac{\pi}{2 \ln(2)} = \frac{\pi}{\ln(4)} \approx 2.2661800710 \quad (7)$$

4. The Coordinate Remainder $k(1)$ is given by:

$$k(1) = K(1) - 2^1 = \frac{\pi}{2 \ln(2)} - 2 \approx 0.2661800710 \quad (8)$$

3 Logarithmic Scale Cancellation

Substituting these evaluations into the bridge relation at $s = 1$:

$$\beta(1) = 2^{-1} \cdot K(1) \cdot \eta(1) \quad (9)$$

$$\beta(1) = \frac{1}{2} \left(\frac{\pi}{2 \ln(2)} \right) \ln(2) \quad (10)$$

The $\ln(2)$ term in the numerator (from the alternating series sum $\eta(1)$) cancels with the $\ln(2)$ term in the denominator (generated by the derivative of the $(2^s - 2)$ prefactor at $s = 1$):

$$\beta(1) = \frac{\pi}{4} \left(\frac{\ln(2)}{\ln(2)} \right) = \frac{\pi}{4} \quad (11)$$

Thus, $\pi/4$ represents the balanced state of the $k(s)$ coordinate bridge where logarithmic dilation from the alternating series is neutralized by the inverse scale of the pole prefactor.