

A Modular Parity Decomposition of the Riemann Zeta Function and Its Asymptotic Floor Boundary

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Abstract

We present a modular parity decomposition of the Riemann Zeta function partitioned modulo 4 into its constituent Hurwitz sub-series. By analyzing the parity sums, alternating series differences, and cross-parity ratios, we isolate the transcendental complexity of the system into a single parameter-free coordinate remainder, $k(n)$. We establish that $k(n)$ converges asymptotically to 1 within an exponential bound contracting at the rate of $(2/3)^n$, and analyze the floor boundary conjecture $\lfloor \zeta(n, 3/4)/\zeta(n) \rfloor = \lfloor (4/3)^n \rfloor$ for $n \geq 2$, connecting discrete Diophantine boundaries to continuous modular L-functions. We provide the complete three-sector analytic coordinate formulas mapping the discrete algebraic scaffold directly to classical Dirichlet and Hurwitz series.

1 Modular Parity Decomposition and Analytic Foundations

The Riemann Zeta function $\zeta(n)$ decomposes modulo 4 into four constituent sub-series indexed by the residue class of the summation variable, formulated via the Hurwitz zeta function $\zeta(n, a)$:

$$4^{-n}\zeta(n, 1/4) = \sum_{k=0}^{\infty} (4k+1)^{-n} \quad (1)$$

$$4^{-n}\zeta(n, 2/4) = \sum_{k=0}^{\infty} (4k+2)^{-n} = 2^{-n}(1 - 2^{-n})\zeta(n) \quad (2)$$

$$4^{-n}\zeta(n, 3/4) = \sum_{k=0}^{\infty} (4k+3)^{-n} \quad (3)$$

$$4^{-n}\zeta(n, 4/4) = \sum_{k=0}^{\infty} (4k+4)^{-n} = 4^{-n}\zeta(n) \quad (4)$$

Linear combinations of these modular sub-series yield the parity direct sums and alternating series:

$$\text{Even Parity Sum} = 4^{-n}\zeta(n, 2/4) + 4^{-n}\zeta(n, 4/4) = 2^{-n}\zeta(n) \quad (5)$$

$$\text{Odd Parity Sum} = 4^{-n}\zeta(n, 1/4) + 4^{-n}\zeta(n, 3/4) = (1 - 2^{-n})\zeta(n) = \frac{1}{2}(\zeta(n) + \eta(n)) \quad (6)$$

$$\text{Even Parity Alternating Sign} = 4^{-n}\zeta(n, 2/4) - 4^{-n}\zeta(n, 4/4) = 2^{-n}\eta(n) \quad (7)$$

$$\text{Odd Parity Alternating Sign} = 4^{-n}\zeta(n, 1/4) - 4^{-n}\zeta(n, 3/4) = \beta(n) = \frac{1}{2}(\beta(n) + \beta(n)) \quad (8)$$

$$\text{Sum Ratio} = \frac{(1 - 2^{-n})\zeta(n)}{2^{-n}\zeta(n)} = 2^n - 1 \quad (9)$$

$$\text{Alternating Sign Ratio} = \frac{\beta(n)}{2^{-n}\eta(n)} = 2^n \frac{\beta(n)}{\eta(n)} = K(n) \quad (10)$$

where $\eta(n)$ is the Dirichlet Eta function and $\beta(n)$ is the Dirichlet Beta function. Subtracting 2^n from $K(n)$ defines the parameter-free coordinate remainder $k(n)$:

$$k(n) = K(n) - 2^n = 2^n \left(\frac{\beta(n)}{\eta(n)} - 1 \right) \quad (11)$$

The coordinate remainder $k(n)$ isolates the non-rational coordinate displacement of the odd-parity Dirichlet sub-series relative to the even-parity baseline.

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2 Three-Sector Analytic Coordinate Verification Matrix

Even Parity ζ		Parent View	Odd Parity ζ	
$\frac{1}{2} \cdot 4^{-n}(\zeta(n, 2/4) + \zeta(n, 4/4))$	$\frac{1}{2} \cdot 4^{-n}(\zeta(n, 2/4) - \zeta(n, 4/4))$	$\frac{1}{2}\zeta$	$\frac{1}{4}(\zeta + \eta)$	$\frac{1}{4}(\beta + \beta)$
$4^{-n}\zeta(n, 2/4)$	$4^{-n}\zeta(n, 4/4)$	$\frac{1}{2}(\zeta + \eta)$	$\frac{1}{4}((\zeta + \eta) + 2\beta)$	$\frac{1}{4}((\zeta + \eta) - 2\beta)$
$4^{-n}(\zeta(n, 2/4) + \zeta(n, 4/4))$	$4^{-n}(\zeta(n, 2/4) - \zeta(n, 4/4))$	ζ	$\frac{1}{2}(\zeta + \eta)$	$\frac{1}{2}(\beta + \beta)$

3 The $\beta - \eta$ Verification Sector

Subtracting the alternating Dirichlet series $\eta(n)$ from the Dirichlet Beta series $\beta(n)$ isolates the parameter-free coordinate remainder $k(n)$ within the odd-parity partitions:

Odd Parity $\beta - \eta$ Sector	
$\frac{1}{4}(\zeta - \eta)$	$\frac{1}{4} \cdot 2(\beta - \eta)$
$\frac{1}{4}((\zeta - \eta) + 2(\beta - \eta))$	$\frac{1}{4}((\zeta - \eta) - 2(\beta - \eta))$
$\frac{1}{2}(\zeta - \eta)$	$\beta - \eta$

4 Coordinate Definitions and Odd Parity Representation

From the $\beta - \eta$ verification sector, we define coordinates $X(n)$ and $Y(n)$:

$$X(n) = \frac{1}{4}(\zeta(n) - \eta(n)) = \frac{1}{2} \cdot 2^{-n}\zeta(n) \quad (12)$$

$$Y(n) = \frac{1}{2}(\beta(n) - \eta(n)) \quad (13)$$

$$r(n) = \frac{Y(n)}{X(n)} = \frac{\frac{1}{2}(\beta(n) - \eta(n))}{\frac{1}{4}(\zeta(n) - \eta(n))} = (1 - 2^{1-n})k(n) \quad (14)$$

$$4^{-n}\zeta(n, 3/4) = X(n) - Y(n) = X(n)(1 - r(n)) \quad (15)$$

5 Asymptotic Convergence and Exponential Bounds

Expanding $\beta(n)$ and $\eta(n)$ in their classical Dirichlet series:

$$\beta(n) = 1 - 3^{-n} + O(5^{-n}), \quad \eta(n) = 1 - 2^{-n} + 3^{-n} + O(4^{-n})$$

The series expansion of their ratio yields:

$$\frac{\beta(n)}{\eta(n)} = 1 + 2^{-n} - 2 \cdot 3^{-n} + O(4^{-n})$$

Substituting into $k(n)$:

$$k(n) = 2^n \left(\frac{\beta(n)}{\eta(n)} - 1 \right) = 1 - 2 \left(\frac{2}{3} \right)^n + O(2^{-n})$$

Consequently, $\lim_{n \rightarrow \infty} k(n) = 1$ and the remainder satisfies the exponential bound:

$$|k(n) - 1| \leq 2 \left(\frac{2}{3} \right)^n \quad \text{for all } n \geq 2 \quad (16)$$

6 Asymptotic Analysis and Floor Boundary Conjecture

Define the normalized coordinate ratio $A(n) = \frac{\zeta(n, 3/4)}{\zeta(n)}$. The difference between $(4/3)^n$ and $A(n)$ is:

$$\delta(n) = \left(\frac{4}{3} \right)^n - A(n) = \frac{1}{\zeta(n)} \sum_{k=1}^{\infty} \left[\left(\frac{4}{3k+3} \right)^n - \left(\frac{4}{4k+3} \right)^n \right] \quad (17)$$

Since $4k+3 > 3k+3 > 0$ for all integers $k \geq 1$, every term in the sum is strictly positive, establishing $\delta(n) > 0$ and $A(n) < (4/3)^n$ for all $n \geq 1$. The leading term at $k = 1$ yields $\delta(n) = \Theta((2/3)^n)$.

n	$(4/3)^n$	Distance to Floor	$A(n) = \zeta(n, 3/4)/\zeta(n)$	Difference $\delta(n)$	$\lfloor A(n) \rfloor$	$\lfloor (4/3)^n \rfloor$
2	1.77777778	0.77777778 (7/9)	1.54527753	0.23250025	1	1
3	2.37037037	0.37037037 (10/27)	2.20564983	0.16472054	2	2
4	3.16049383	0.16049383 (13/81)	3.07689108	0.08360275	3	3
5	4.21399177	0.21399177 (52/243)	4.17523992	0.03875185	4	4
6	5.61865569	0.61865569 (451/729)	5.60100416	0.01765153	5	5

Letting $\{x\} = x - \lfloor x \rfloor$ denote the fractional part, the condition for the floor identity $\lfloor A(n) \rfloor = \lfloor (4/3)^n \rfloor$ to hold is equivalent to the Diophantine inequality:

$$\left\{ \left(\frac{4}{3} \right)^n \right\} > \delta(n) \quad \text{for all } n \geq 2 \quad (18)$$

Conjecture 1 (Floor Boundary Conjecture). *For all integers $n \geq 2$:*

$$\lfloor A(n) \rfloor = \left\lfloor \frac{\zeta(n, 3/4)}{\zeta(n)} \right\rfloor = \left\lfloor \left(\frac{4}{3} \right)^n \right\rfloor \quad (19)$$

This relationship has been verified computationally for all integers $2 \leq n \leq 50,000$.

7 Modular Parity Partition Convergence

8 Discussion and Future Directions

The modulo-4 partition separates the Riemann Zeta function into an even-parity sector whose sub-series evaluate to rational multiples of $\zeta(n)$ and $\eta(n)$, and an odd-parity sector parameterized by the coordinate remainder $k(n) = 2^n(\beta(n)/\eta(n) - 1)$. The exponential contraction $|k(n) - 1| \leq 2(2/3)^n$ establishes that the odd-parity sub-series asymptotically recover the structural symmetry of the even-parity sector.

Direct avenues of research include investigating the algebraic structure of $(2^n - 2)k(n)$ across odd integer arguments, extending coordinate definitions $X(s), Y(s)$ into the critical strip, and generalizing modular partitions to higher bases modulo $2m$.

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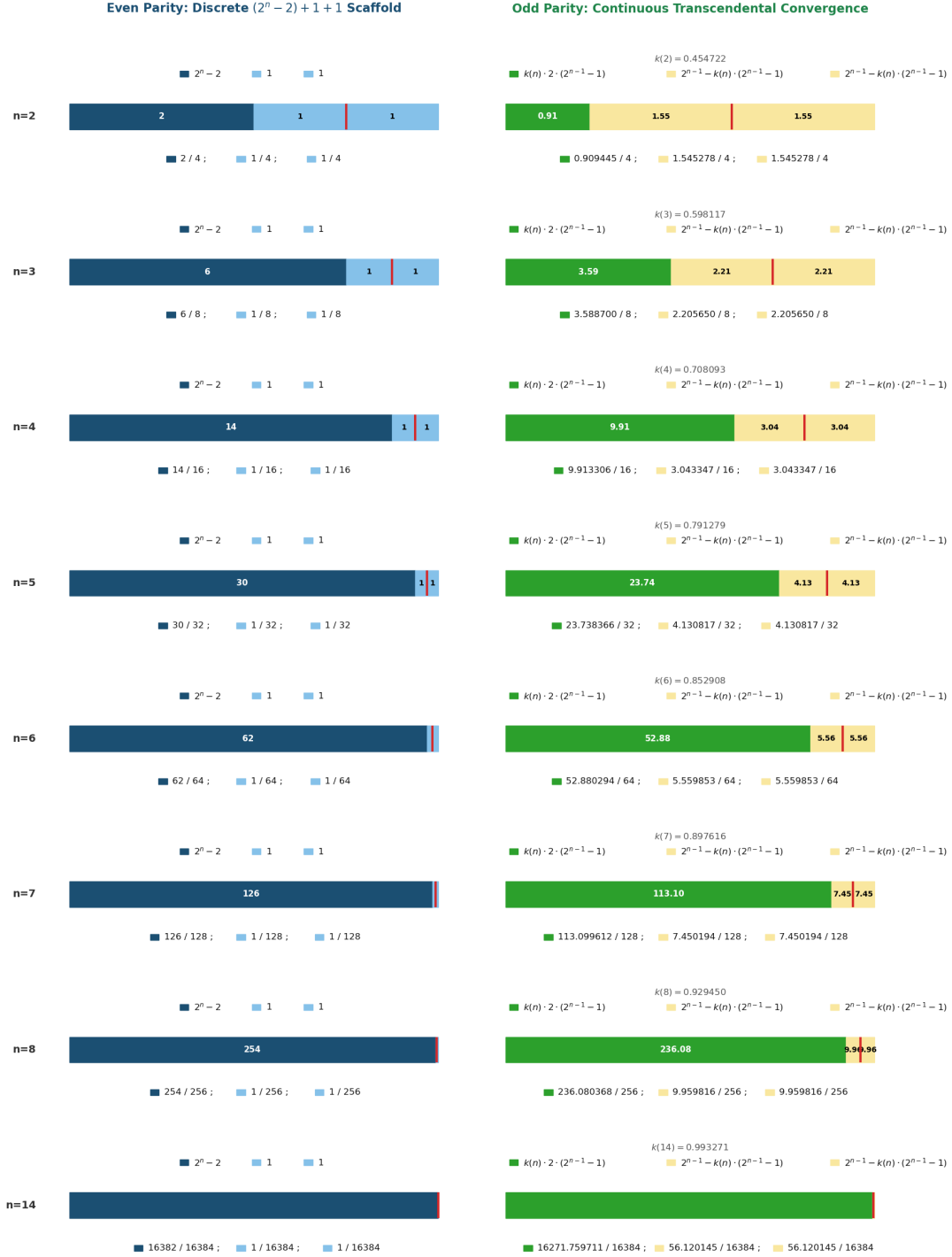


Figure 1: Evolution and partition of unity for the four modular Hurwitz sub-series $4^{-n}\zeta(n, a/4)/\zeta(n)$ for $a \in \{1, 2, 3, 4\}$ across integer arguments $n = 2$ to 8 and $n = 14$.